Multiple Choice

1 D) $a=1, b=4, c=-1$ (this is the first time in HSC, they only give 1 mark for finding a,b,c. They clearly want you to learn my short cut method)

2 A) $S=4, P=5, \therefore x^2-4x+5$

3 B) $b^2=a^2(1-e^2), \therefore e^2=1-\dfrac{b^2}{a^2}=1-\dfrac{\frac{25}{16}}{\frac{25}{9}}=1-\dfrac{9}{16}$

$=\dfrac{7}{16}, \therefore e=\dfrac{\sqrt{7}}{4}$

4 C) $\dfrac{1}{\bar{z}}=\dfrac{1}{2\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)}=\dfrac{1}{2}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

5 C) $y=\pm\sqrt{x^2-2x}$

6 D) Radius $=2-x, \therefore V=2\pi\int_0^4 (2-x)^2\,dy$

7 B) $\displaystyle\int \frac{1}{1-\sin x}dx=\int\frac{1+\sin x\,dx}{\cos^2 x}$

$=\displaystyle\int(\sec^2 x+\sec x\tan x)dx=\tan x+\sec x+C$

8 B) $w+u=z$ (parallelogram method), and $\arg(w)+\arg(z)=\arg(u)$

9 B) A is not, because $a=2\cos(x-1), \therefore$ initially $a=2$, C is not because $x\neq 1$, D is not because $x=1, v\neq 2$

10 D) $\displaystyle\int_0^a f(x)dx=\int_0^a f(a-x)dx$, using $u=a-x$

and $\displaystyle\int_{-a}^0 f(x)dx=\int_0^a f(-x)dx$, using $u=-x$

$=\displaystyle\int_0^a f(x-a)dx$, using $u-a=-x$

Question 11

(a) (i) $z+w=1-i=\sqrt{2}\,\text{cis}\left(-\dfrac{\pi}{4}\right)$

(ii) $\dfrac{z}{w}=\dfrac{-2-2i}{3+i}=\dfrac{(-2-2i)(3-i)}{10}=\dfrac{-8-4i}{10}$

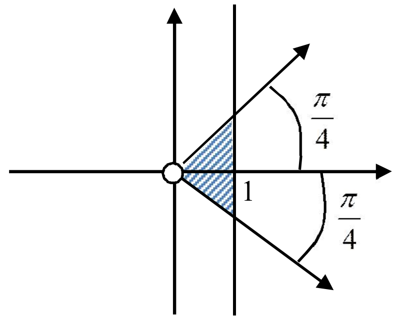
$=\dfrac{-4-2i}{5}$

(b) Let $u=3x-1, du=3dx;\ dv=\cos(\pi x)dx, v=\dfrac{\sin(\pi x)}{\pi}$

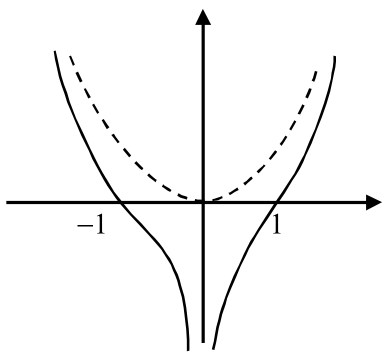
$\displaystyle\int_0^{\frac{1}{2}}(3x-1)\cos(\pi x)dx=\left[\frac{(3x-1)\sin(\pi x)}{\pi}\right]_0^{\frac{1}{2}}$

$\displaystyle-\frac{3}{\pi}\int_0^{\frac{1}{2}}\sin(\pi x)dx=\frac{1}{2\pi}+\frac{3}{\pi^2}\Big[\cos(\pi x)\Big]_0^{\frac{1}{2}}=\frac{1}{2\pi}-\frac{3}{\pi^2}$

(c)



(d)



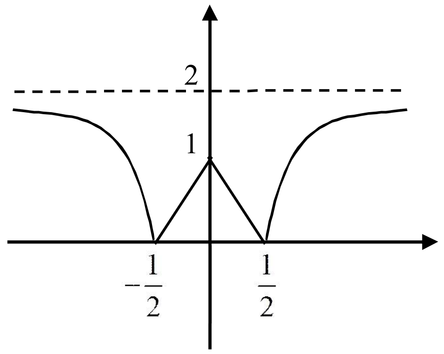
(e) $\partial V=2\pi rh\partial y=2\pi y^2(6-y)\partial y=2\pi(6y^2-y^3)\partial y$

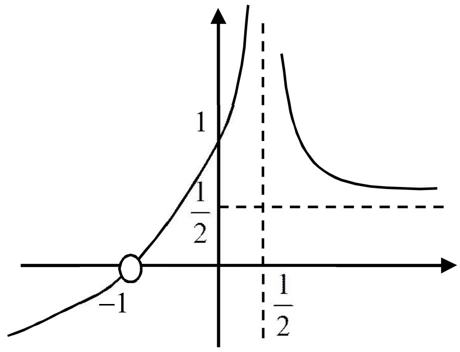
$V=2\pi\displaystyle\int_0^6(6y^2-y^3)dy=2\pi\left[2y^3-\frac{y^4}{4}\right]_0^6$

$=2\pi(432-324)=216\pi$ u^3.

Question 12

(a)





(b) (i) Sub. $x = 2\cos\theta$,

$8\cos^3\theta - 6\cos\theta = \sqrt{3}$

$2\cos 3\theta = \sqrt{3}$

$\cos 3\theta = \frac{\sqrt{3}}{2}$

(ii) Solving $\cos 3\theta = \frac{\sqrt{3}}{2}$ gives $3\theta = \pm\frac{\pi}{6} + k2\pi, k \in Z$

$\therefore \theta = \pm\frac{\pi}{18} + \frac{k2\pi}{3}$

$\therefore x = 2\cos\frac{\pi}{18}, 2\cos\frac{11\pi}{18}, 2\cos\frac{13\pi}{18}.$

(c) For $x^2 - y^2 = 5, 2x - 2yy' = 0, \therefore y' = \frac{x}{y}.$

$\therefore$ At $(x_0, y_0), m_1 = \frac{x_0}{y_0}.$

For $xy = 6, y + xy' = 0, \therefore y' = -\frac{y}{x}.$

$\therefore$ At $(x_0, y_0), m_2 = -\frac{y_0}{x_0}.$

$m_1 m_2 = -1, \therefore$ the tangents are perpendicular

(d) (i) $I_0 = \int_0^1 \frac{1}{x^2+1}dx = \left[\tan^{-1}x\right]_0^1 = \frac{\pi}{4}$

(ii) $I_n + I_{n-1} = \int_0^1 \frac{x^{2n} + x^{2n-2}}{x^2+1}dx = \int_0^1 x^{2n-2}dx$

$= \left[\frac{x^{2n-1}}{2n-1}\right]_0^1 = \frac{1}{2n-1}$

(iii) $\int_0^1 \frac{x^4}{x^2+1} = I_2$

$I_2 + I_1 = \frac{1}{3}$

$I_1 + I_0 = \frac{1}{1} = 1$

$\therefore I_2 = \frac{1}{3} - 1 + \frac{\pi}{4} = \frac{\pi}{4} - \frac{2}{3}$

Question 13

(a) Let $t=\tan\frac{x}{2}, dt=\frac{1}{2}\sec^2\frac{x}{2}dx, \therefore dx=\frac{2dt}{1+t^2}$

When $x=\frac{\pi}{2}, t=1$; When $x=\frac{\pi}{3}, t=\frac{1}{\sqrt{3}}$

$$I=\int_{\frac{1}{\sqrt{3}}}^{1}\frac{1}{\frac{6t}{1+t^2}-\frac{4-4t^2}{1+t^2}+5}\frac{2dt}{1+t^2}$$

$$=\int_{\frac{1}{\sqrt{3}}}^{1}\frac{2dt}{6t-4+4t^2+5+5t^2}$$

$$=\int_{\frac{1}{\sqrt{3}}}^{1}\frac{2dt}{9t^2+6t+1}=\int_{\frac{1}{\sqrt{3}}}^{1}\frac{2dt}{(3t+1)^2}=\left[\frac{-2}{3(3t+1)}\right]_{\frac{1}{\sqrt{3}}}^{1}$$

$$=\frac{2}{3}\left(\frac{1}{\sqrt{3}+1}-\frac{1}{4}\right)=\frac{2}{3}\left(\frac{3-\sqrt{3}}{4(\sqrt{3}+1)}\right)=\frac{1}{6}\left(\frac{(3-\sqrt{3})(\sqrt{3}-1)}{2}\right)$$

$$=\frac{3\sqrt{3}-3-3+\sqrt{3}}{12}=\frac{4\sqrt{3}-6}{12}=\frac{2\sqrt{3}-3}{6}.$$

(b) The trapezium has parallel sides y and $2y$, and height $\frac{\sqrt{3}y}{2}$.

The area of the trapezium $=\frac{\sqrt{3}y}{4}(y+2y)=\frac{3\sqrt{3}}{4}y^2$.

$$\partial V=\frac{3\sqrt{3}}{4}y^2\partial x=\frac{3\sqrt{3}}{4}x^4\partial x$$

$$V=\frac{3\sqrt{3}}{4}\int_0^2 x^4dx=\frac{3\sqrt{3}}{4}\left[\frac{x^5}{5}\right]_0^2=\frac{24\sqrt{3}}{5}\text{ u}^3.$$

(c) (i) Sub to the hyperbola

$$\text{LHS}=\frac{(t^2+1)^2}{4t^2}-\frac{(t^2-1)^2}{4t^2}=\frac{t^4+2t^2+1-t^4+2t^2-1}{4t^2}$$

$=\frac{4t^2}{4t^2}=1=\text{RHS}, \therefore M$ belongs to the hyperbola

(ii) $$m_{PQ}=\frac{bt+\frac{b}{t}}{at-\frac{a}{t}}=\frac{b(t^2+1)}{a(t^2-1)}$$

$$m_M=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\frac{b(4t^2-2(t^2-1))}{4t^2}}{\frac{a(4t^2-2(t^2+1))}{4t^2}}=\frac{b(2t^2+2)}{a(2t^2-2)}=\frac{b(t^2+1)}{a(t^2-1)}$$

$=m_{PQ}, \therefore PQ$ is the tangent.

(iii) $$OP\times OQ=\sqrt{(at)^2+(bt)^2}\sqrt{\frac{a^2}{t^2}+\frac{b^2}{t^2}}=a^2+b^2$$

$$=a^2+a^2(e^2-1)=a^2e^2=OS^2$$

(iv) If $x_P=x_S, \therefore at=ae, \therefore t=e$

$$m_{MS}=\frac{\frac{b(e^2-1)}{2e}}{\frac{a(e^2+1)}{2e}-ae}=\frac{b(e^2-1)}{a(1-e^2)}=-\frac{b}{a}=$$ gradient of one asymptote, $\therefore MS$ is parallel to one asymptote

Question 14

(a) (i) $P(x)=x^5-10x^2+15x-6$

$P'(x)=5x^4-20x+15$

$P''(x)=20x^3-20$

$P(1)=P'(1)=P''(1)=0, \therefore 1$ is the triple root

(ii) $\sum\alpha=3+\alpha+\beta=0, \therefore \alpha+\beta=-3$

$\prod\alpha=\alpha\beta=6$

$\therefore \alpha$ and β satisfy $x^2+3x+6=0$.

$\therefore$ The other 2 roots are $\dfrac{-3\pm\sqrt{-15}}{2}=\dfrac{-3\pm\sqrt{15}i}{2}$

(b) (i) $m_{OP}=m_1=\dfrac{b\sin\theta}{a\cos\theta}=\dfrac{b}{a}\tan\theta$

Gradient of the normal $m_2=\dfrac{-\frac{dx}{d\theta}}{\frac{dy}{d\theta}}=\dfrac{a\sin\theta}{b\cos\theta}=\dfrac{a}{b}\tan\theta$

$\therefore \tan\phi=\dfrac{\left(\frac{a}{b}-\frac{b}{a}\right)\tan\theta}{1+\tan^2\theta}=\dfrac{a^2-b^2}{ab}\dfrac{\tan\theta}{\sec^2\theta}$

$=\dfrac{a^2-b^2}{ab}\sin\theta\cos\theta$

(ii) $\tan\phi=\dfrac{a^2-b^2}{2ab}\sin 2\theta$

$\therefore \phi$ is maximum when $\sin 2\theta=\pm1, \therefore \theta=\pm\dfrac{\pi}{4},\pm\dfrac{3\pi}{4}$

(c) (i) $m\ddot{x}=F-Kv^2$

The terminal velocity occurs when $\ddot{x}=0$,

$\therefore F-K\times300^2=0, \therefore K=\dfrac{F}{300^2}$.

$\therefore m\ddot{x}=F\left(1-\left(\dfrac{v}{300}\right)^2\right)$.

(ii) $m\dfrac{dv}{dt}=F\left(1-\left(\dfrac{v}{300}\right)^2\right)$.

$$\frac{dv}{1-\left(\frac{v}{300}\right)^2}=\frac{F}{m}dt$$

$$\int_0^{200}\frac{dv}{300^2-v^2}=\frac{F}{300^2m}\int_0^T dt$$

$$\text{LHS}=\int_0^{200}\frac{dv}{(300-v)(300+v)}$$

$$=\frac{1}{600}\int_0^{200}\left(\frac{1}{300-v}+\frac{1}{300+v}\right)dv$$

$$=\frac{1}{600}\left[\ln\frac{300+v}{300-v}\right]_0^{200}=\frac{1}{600}\ln 5$$

$$\therefore \frac{FT}{300^2m}=\frac{1}{600}\ln 5$$

$\therefore T=\dfrac{150m\ln 5}{F}$ hours.

Question 15

(a) $1=(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc$

But $(a-b)^2\ge 0, \therefore a^2+b^2\ge 2ab\ge 2a^2$, since $a\le b$

Similarly, $a^2+c^2\ge 2ac\ge 2a^2$ and $b^2+c^2\ge 2bc\ge 2b^2$, since $a\le b\le c$.

$\therefore 1=a^2+b^2+c^2+2ab+2ac+2bc$

$\ge a^2+b^2+c^2+2a^2+2a^2+2b^2$

$=5a^2+3b^2+c^2$.

(b) (i) $1+i=\sqrt{2}\operatorname{cis}\frac{\pi}{4}$,

$\therefore (1+i)^n=\left(\sqrt{2}\right)^n\operatorname{cis}\frac{n\pi}{4}=\left(\sqrt{2}\right)^n\left(\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}\right)$

$1-i=\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$,

$\therefore (1-i)^n=\left(\sqrt{2}\right)^n\operatorname{cis}\left(-\frac{n\pi}{4}\right)$

$=\left(\sqrt{2}\right)^n\left(\cos\left(-\frac{n\pi}{4}\right)+i\sin\left(-\frac{n\pi}{4}\right)\right)$

$=\left(\sqrt{2}\right)^n\left(\cos\frac{n\pi}{4}-i\sin\frac{n\pi}{4}\right)$, since $\cos x$ is even, $\sin x$ is odd.

$\therefore (1+i)^n+(1-i)^n=2\left(\sqrt{2}\right)^n\cos\frac{n\pi}{4}$

(ii) $(1+i)^n=\binom{n}{0}+i\binom{n}{1}-\binom{n}{2}-i\binom{n}{3}+\binom{n}{4}+\ldots$

$(1-i)^n=\binom{n}{0}-i\binom{n}{1}-\binom{n}{2}+i\binom{n}{3}+\binom{n}{4}+\ldots$

If n is a multiple of 4, the last term in each of the above expressions is $+\binom{n}{n}$.

$\therefore \frac{(1+i)^n+(1-i)^n}{2}=\binom{n}{0}-\binom{n}{2}+\binom{n}{4}-\ldots+\binom{n}{n}$.

$\therefore \binom{n}{0}-\binom{n}{2}+\binom{n}{4}-\ldots+\binom{n}{n}=\left(\sqrt{2}\right)^n\cos\frac{n\pi}{4}$

If n is a multiple of 4, let $n=4k, \cos\frac{n\pi}{4}=\cos k\pi$

$=1$ if k is even, or -1 if k is odd.

$\therefore \cos\frac{n\pi}{4}=(-1)^k=(-1)^{\frac{n}{4}}$.

$\therefore \binom{n}{0}-\binom{n}{2}+\binom{n}{4}-\ldots+\binom{n}{n}=(-1)^{\frac{n}{4}}\left(\sqrt{2}\right)^n$.

(c) (i) Resolving the forces

vertically, $kv^2=mg+T\sin\phi$ (1)

horizontally, $T\cos\phi=\frac{mv^2}{r}=\frac{mv^2}{\ell\cos\phi}$ (2)

(1) gives $\sin\phi=\frac{kv^2-mg}{T}$ (3)

(2) gives $\cos^2\phi=\frac{mv^2}{T\ell}$ (4)

$\frac{(3)}{(4)}=\frac{\sin\phi}{\cos^2\phi}=\frac{\ell kv^2-\ell mg}{mv^2}=\frac{\ell k}{m}-\frac{\ell g}{v^2}$.

(ii) $\sin\phi<\frac{\ell k}{m}\cos^2\phi=\frac{\ell k}{m}\left(1-\sin^2\phi\right)$.

$\sin^2\phi+\frac{m}{\ell k}\sin\phi-1<0$

$\therefore \sin\phi<\frac{-\frac{m}{\ell k}+\sqrt{\frac{m^2}{\ell^2k^2}+4}}{2}=\frac{\sqrt{m^2+4\ell^2k^2}-m}{2\ell k}$.

(iii) Let $f(\phi)=\frac{\sin\phi}{\cos^2\varphi}=\sec\phi\tan\phi$.

$f'(\phi)=\sec\phi\tan^2\phi+\sec^3\phi=\sec\phi(\tan^2\phi+\sec^2\phi)$

For $-\frac{\pi}{2}<\phi<\frac{\pi}{2}, \sec\phi>0, \therefore f'(\phi)>0, \therefore f(\phi)$ is increasing.

(iv) When ϕ increases, $\frac{\sin\phi}{\cos^2\phi}$ increases, $\therefore \frac{\ell g}{v^2}$ decreases (since $\frac{\ell k}{m}$ is a constant), $\therefore v^2$ increases

Question 16

(a) (i) $\angle APX = \angle ADP$ (angles in alternate segments)

$\angle ADP = \angle DPQ$ (alternate angles on parallel lines)

$\therefore \angle APX = \angle DPQ$.

(ii) Similarly, $\angle RPC = \angle YPB$

$\angle QPR = \angle XPY$ (vertically opposite)

$\angle APD = \angle BPC = 90°$ (semi-circle angles)

$\therefore$ Let $\angle APX = \angle DPQ = \alpha, \angle RPC = \angle YPB = \beta$,

and $\angle QPR = \angle XPY = \gamma$

$\angle APD + \angle DPQ + \angle QPR + \angle RPC + \angle CPB + \angle BPY$

$+ \angle YPX + \angle XPA = 360°$ (angle at a point)

$\therefore 2\alpha + 2\beta + 2\gamma + 180° = 360°$

$\therefore \alpha + \beta + \gamma = 90°$

$\therefore \angle APC = 90° + 90° = 180°$, i.e. A, P, C are collinear.

(iii) Since A, P, C are collinear, $\angle RPC = \angle APX$

(vertically opposite), $\therefore \alpha = \beta$.

$\therefore ABCD$ is a cyclic quadrilateral (angles subtending the same arc are equal)

(b) (i) $1 - x^2 + x^4 - ... + (-1)^{n-1} x^{2n-2}$ is a GP,

$$\therefore \text{Sum} = \frac{1-(-x^2)^n}{1-(-x^2)} = \frac{1-(-x^2)^n}{1+x^2}.$$

$$\frac{1}{1+x^2} - \frac{1-(-x^2)^n}{1+x^2} = \frac{(-x^2)^n}{1+x^2}$$

Since $-x^{2n} \le \frac{(-x^2)^n}{1+x^2} \le x^{2n}$, we conclude that

$$-x^{2n} \le \frac{1}{1+x^2} - \left(1 - x^2 + x^4 - ... + (-1)^{n-1} x^{2n-2}\right) \le x^{2n}$$

(ii) Integrating wrt x between 0 and 1,

$$\left[\frac{-x^{2n+1}}{2n+1}\right]_0^1$$

$$\le \left[\tan^{-1} x - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - ... + \frac{(-1)^{n-1} x^{2n-1}}{2n-1}\right)\right]_0^1$$

$$\le \left[\frac{x^{2n+1}}{2n+1}\right]_0^1$$

$$\frac{-1}{2n+1} \le \frac{\pi}{4} - \left(1 - \frac{1}{3} + \frac{1}{5} - ... + (-1)^{n-1}\frac{1}{2n-1}\right) \le \frac{1}{2n+1}$$

(iii) Let $n \to \infty, \frac{\pm 1}{2n+1} \to 0$,

$$\therefore \frac{\pi}{4} - \lim_{n\to\infty}\left(1 - \frac{1}{3} + \frac{1}{5} - ... + ... + (-1)^{n-1}\frac{1}{2n-1}\right) \to 0$$

$$\therefore \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - ...$$

(c) $\int \frac{\ln x}{(1+\ln x)^2} dx = \int \frac{x \ln x}{x(1+\ln x)^2} dx.$

By IBP, let $u = x\ln x, du = \ln x + 1$

$$dv = \frac{1}{x(1+\ln x)^2} dx, v = \frac{-1}{1+\ln x}$$

$$I = \frac{-x\ln x}{1+\ln x} + \int \frac{\ln x + 1}{1+\ln x} dx$$

$$= \frac{-x\ln x}{1+\ln x} + x + C$$

$$= \frac{x}{1+\ln x} + C$$